The many faces of entropy

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Let's begin!! A quite famous anecdote ...

The Neumann-Shannon anecdote[1] has been retold so many times that it has classified by some as an urban legend in science.

Myron Tribus, an American engineer, in one of his 1987 articles, recounts his discussion with Shannon about the name "entropy", as follows:

"The same function appears in statistical mechanics and, on the advice of John von Neumann, Claude Shannon called it "entropy". I talked with Dr. Shannon once about this, asking him why he had called his function by a name that was already in use in another field. I said that it was bound to cause some confusion between the theory of information and thermodynamics. He said that Von Neumann had told him: "No one really understands entropy. Therefore, if you know what you mean by it and you use it when you are in an argument, you will win every time."

In the "Introduction" of David Goodstein's book "States of Matter" a warning for the reader appears, which is short of black humor (link on bookmarks bar) ...

[1] https://www.eoht.info/page/Neumann-Shannon/anecdote

So what about the name "entropy" then ...

Is there any connection to the Greek words "$\nu\tau\rho o\pi\eta$" or "$\nu\tau\rho o\pi\alpha\lambda\eta$" which, the second one, since it is a word of feminine gender, means: "shy girl" in English? ;-)

Of course not!!!!

With its Greek prefix en-, meaning "within", and the trop- root here meaning "change", entropy basically means "change within (a closed system)".

According to American scholar Eric Zencey[2]: "Clausius coined the term entropy, from the Greek tropos, or transformation."

According to Ilya Prigogine, in the "End of Certainty", entropy simply means "evolution" ...

[2] https://www.eoht.info/page/Entropy/(etymology)

Gounaris - definition of temperature

κατάστασις ἰσορροπίας συστήματος

μή ἰσορροπίας.

Ἡ κατάστασις ἰσορροπίας ἑνός συστήματος καθορίζεται πλήρως συναρ
τήσει ὡρισμένων ἀπό τάς χαρακτηριζούσας τό σύστημα παραμέτρους, ὁπότε
αἱ ἄλλαι ὑπολογίζονται ὡς συναρτήσεις αὐτῶν. Μία τοιαύτη συνάρτησις εἶ
ναι π.χ.

$$T = f(P,V,M,\alpha_1,\alpha_2, \ldots ,\alpha_n) \qquad (2.$$

ἔνθα T = θερμοκρασία, P = πίεσις, V = ὄγκος, M = μᾶζα τοῦ συστήματος κα
$\alpha_1,\alpha_2, \ldots ,\alpha_n$ παριστάνουν ὅλας τάς ἄλλας παραμέτρους ἐκ τῶν ὁποίων δυ
ναται νά ἐξαρτᾶται τό σύστημα. Ἡ (2.1) λέγεται <u>ἐξίσωσις καταστάσεως</u> τ
συστήματος. Παρατηροῦμεν ὅτι α) Εἰς τήν (2.1) ἐδώσαμεν μίαν γενικήν μο
φήν τῆς ἐξισώσεως καταστάσεως. Εἶναι δυνατόν ὅμως τό σύστημά μας νά ε
ναι τοιοῦτον ὥστε π.χ. ἡ πίεσις νά μήν ἐμφανίζεται εἰς τό δεξιόν μέρ
τῆς (2.1)· εἰς τήν περίπτωσιν αὐτήν ἡ θερμοκρασία εἶναι ἀνεξάρτητος τῆ
πιέσεως. β) Παρατηρήσατε ὅτι ἡ (2.1) δέν θά ἴσχυε διά συστήματα παρου
άζοντα τό φαινόμενον τῆς ὑστερήσεως κατά τόν

Gounaris - definition of entropy

(E,V,N,ξ) αἱ τιμαί τῶν παραμέτρων ξ μεταβάλλονται μέχρις ὅτου λάβου
τάς τιμάς τάς ἀντιστοιχούσας εἰς τήν πλέον πιθανήν κατάστασιν ἡ ὁπο
ξ ὁρισμοῦ εἶναι ἡ κατάστασις ἰσορροπίας. Ἐφ᾽ὅσον τό σῶμα εὑρεθῆ ε
ήν κατάστασιν τῆς ἰσορροπίας καί δέν διαταραχθῆ ἔξωθεν παραμένει ε
ὐτήν. Διακυμάνσεις, ἔστω καί πολύ μικραί, ἐκ τῆς θέσεως ἰσορροπίας
χουν σχεδόν μηδενικήν πιθανότητα ἐμφανίσεως. Τοῦτο σημαίνει ὅτι τ
ιστικόν βάρος εἶναι ἀμελητέον διά τιμάς τῶν ξ διαφορετικάς τῶν τι
σορροπίας.

Εἰς τήν Στατιστικήν Μηχανικήν (καί τήν θερμοδυναμικήν) χρησι
ῦμεν τήν ἐντροπίαν ἀντί τοῦ στατιστικοῦ βάρους Ω ὡς μέτρον τῆς ἀτ
ῆς μακροκαταστάσεως (E,V,N,ξ). Αὕτη (δι᾽ἕν ἀπομονωμένον σῶμα) ὁρίζ

$$S(E,V,N,\xi) = k\ln\Omega(E,V,N,\xi) \quad (3.$$

Gounaris - definition of entropy as a sum

τότε ἡ ἐντροπία ἑνός συστήματος τῆς

$$S = -k\sum_r p_r \ln p_r \qquad (3.28)$$

Δύναται νά ἀποδειχθῆ ὅτι ὁ ἀνωτέρω ὁρισμός τῆς ἐντροπίας εἶναι ἰσοδύναμος τοῦ (3.3). Πράγματι θεωρήσωμεν ἀπομονωμένον σύστημα μέ ἐνέργειαν E. Ἡ ἀντίστοιχος συλλογή εἶναι ἡ μικροκανονική. Αἱ δυναταί μικροκαταστάσεις τοῦ συστήματος εἶναι αἱ ἔχουσαι ἐνεργείας ἴσας πρός E. Ἡ βασική ὑπόθεσις τῆς Στατιστικῆς Μηχανικῆς (ἰδέ § 3.2) εἶναι ὅτι ὅλαι αὐταί αἱ μικροκαταστάσεις εἶναι ἐξ ἴσου πιθαναί. Ἄρα $p_r = \frac{1}{\Omega(E,V,N)}$ καί ἐκ τῆς (3.28) ἔχομεν

$$S = -k\sum_{r=1}^{\Omega(E,V,N)} \frac{1}{\Omega(E,V,N)} \ln\left(\frac{1}{\Omega(E,V,N)}\right) = k\ln\Omega(E,V,N) \qquad (3.3)$$

Gounaris - maximum of entropy

V_1, V_2, N_1, N_2 παραμένουν σταθερά. Ἐν τοιοῦτον διαχώρισμα καλεῖται διαθερμικόν. Συμφώνως πρός τόν δεύτερον θερμοδυναμικόν νόμον ἡ E_1 μεταβάλλεται καί λαμβάνει τήν τιμήν ἐκείνην διά τήν ὁποίαν ἡ $S(E,V,N,E_1,V_1,N_1)$ ἔχει μέγιστον. Ἔχομεν λοιπόν

$$\frac{\partial S(E,V,N,E_1,V_1,N_1)}{\partial E_1} = \frac{\partial S_1(E_1,V_1,N_1)}{\partial E_1} + \frac{\partial S_2(E_2,V_2,N_2)}{\partial E_2}\,\frac{dE_2}{dE_1} =$$

$$= \frac{\partial S_1(E_1,V_1,N_1)}{\partial E_1} - \frac{\partial S_2(E_2,V_2,N_2)}{\partial E_2} = 0 \quad (3.7)$$

Δι' ἕν ἀπομονωμένον σύστημα ἐν ἰσορροπίᾳ ὁρίζομεν τήν (ἀπόλυτον) θερμοκρασίαν του ἐκ τῆς σχέσεως

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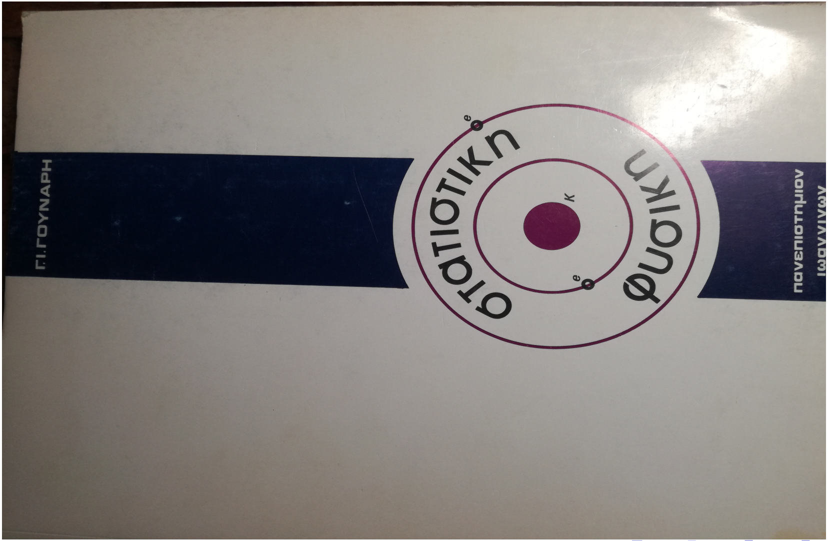
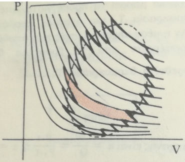
* Διά νά εἶναι ἡ ἐνέργεια ἐκτατική μεταβλητή εἰς ἕν σύστημα θά πρέπει ἡ ἐνέργεια ἐπιφανείας νά εἶναι ἀμελητέα. Αὐτό συμβαίνει συνήθως. Παρομοίου εἴδους προϋπόθεσις ὑφίσταται καί διά τάς ἄλλας ἐκτατικάς μεταβλητάς. Παρατηρήσατε ὅτι αἱ παράμετροι E_1, V_1, N_1 παίζουν τόν ρόλον τῶν παραμέτρων ξ εἰς τήν $S(E,V,N,E_1,V_1,N_1)$.


Γ.Ι.ΓΟΥΝΑΡΗ
στατιστικη
φυσικη
πανεπιστημιον

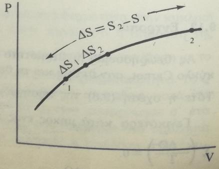
Book of Lyceum - Entropy



9.16. Η εστιγμένη καμπύλη δείχνει έναν τυχαίο αντιστρεπτό που μπορεί να προσεγγιστεί από μια σειρά κύκλων Carnot. πιο πυκνές είναι οι ισόθερμες τόσο πιο καλή είναι η προσέγγιση. αμμοσκιασμένο εμβαδόν παριστάνει έναν τέτοιο στοιχειώδη Carnot.

διαβατικές και ισόθερμες καμπύλες, να προσεγγιστεί με μια Carnot, η τελευταία σχέση ισχύει για κάθε κυκλική αντιστρεπτή

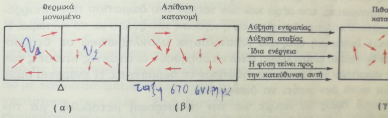
υμε μια οποιαδήποτε αν- βολή του μέσου Μ. Τότε η ντροπίας ΔS κατά τη με- α σημείο 1 σ' ένα άλλο εταβολής, σχ. 9.17, είναι που ΔQ είναι η στοιχειώ- θερμότητας που απορρο- α στοιχειώδες τμήμα της ά το οποίο το μέσον Μ ερμοκρασία Τ. Όπως και ή ενέργεια έτσι και στην οι μεταβολές αυτής έχουν η μεταβολή στην εντρο- σύστημα μεταβαίνει από ισορροπίας σε μια άλλη, εξαρτάται μόνο



Σχ. 9.17

κινούμενα (ταχύτητα

Αφαιρώντας το διάφραγμα το αέριο εμφανίζει την εικόνα του στιγμ Στο στιγμιότυπο γ φαίνεται η τελική κατάσταση του αερίου. Στη από το β στο γ δε μεταβλήθηκε η ενέργεια του αερίου, άλλα κατανομή των μορίων με τις διαφορετικές ταχύτητες στο χώρο το Στο β υπάρχει μια οργάνωση (τάξη) στην κατανομή των ταχυ μορίων, δηλαδή τα μόρια με μεγάλες ταχύτητες είναι συγκεντρ αριστερό μέρος του δοχείου και τα μόρια με μικρές ταχύτητες είν τρωμένα στο δεξιό, ενώ στο στιγμιότυπο γ υπάρχει πλήρης στιγμιότυπο β έχει μικρή πιθανότητα να παρατηρηθεί, ενώ συνήθως παρατηρούμε είναι το στιγμιότυπο γ, το οποίο είναι και το π



Σχ. 9.18

Δηλαδή η εξέλιξη του φαινομένου οδηγεί σε αύξηση της αταξίας και ξη αυτού στην ισορροπία οδηγεί στη μέγιστη αταξία των μορίων.

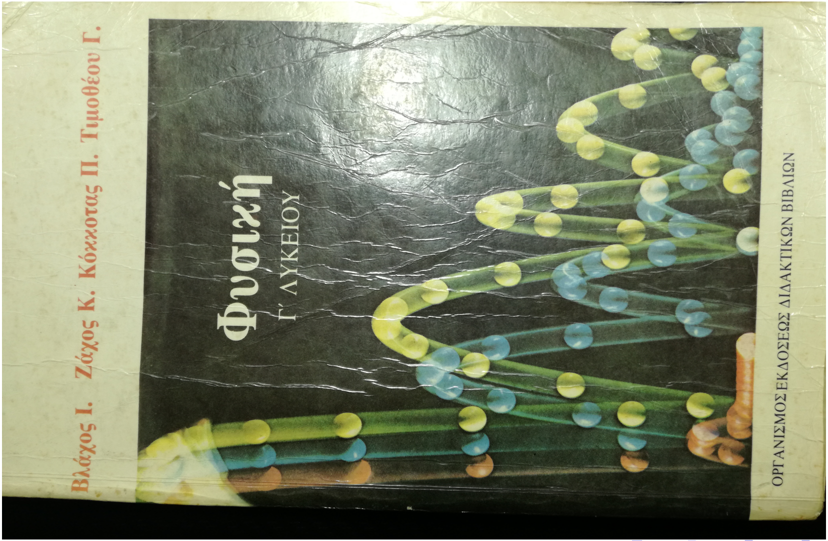
Βλέπουμε λοιπόν ότι κάθε φυσική μεταβολή τείνει σε κατάστασ τερης αταξίας η οποία είναι και η πιο πιθανή.

Αποδεικνύεται ότι η εντροπία S που έχει μια κατάσταση γ, σχ. 9 πιθανότητα P να παρατηρηθεί αυτή η κατάσταση, συνδέονται με

$$S = K \ln P$$

όπου Κ μια σταθερά που ονομάζεται σταθερά Boltzmann. Η εξίσ μας δίνει τη δυνατότητα να αντιληφθούμε ότι η φορά κατά τη γίνονται οι μεταβολές στη φύση είναι από μια κατάσταση β σε κατάσταση γ μεγαλύτερης εντροπίας. Όσο όμως αυξάνει η εντρο συστήματος τόσο λιγότερο διαθέσιμη για έργο γίνεται η ενέργ

Book of Lyceum - Edition 1985



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Thermodynamic and Statistical Entropy

Claussius (Thermodynamic) Entropy

According to the Clausius equality, for a closed homogeneous system, in which only reversible processes take place, the entropy variation is defined as, Entropy S is considered as a state function.

where T the uniform temperature of the closed system and dQ the incremental reversible transfer of heat energy into that system.

$$dS \equiv \frac{dQ}{T} \quad \text{or} \quad S_a - S_b \equiv \int_a^b \frac{dQ}{T}$$

If non reversible processes take place, then the entropy increases

$$\Delta S = S_a - S_b \geq \int_a^b \frac{dQ}{T} \ ,$$

Fundamental thermodynamic relation

$$dU = TdS - PdV$$

Boltzmann (Statistical) Entropy

Ludwig Boltzmann defined entropy, S, as a measure of the number of possible microscopic states (microstates) of a system in thermodynamic equilibrium, consistent with its macroscopic thermodynamic properties, which constitute the macrostate of the system. Therefore the system can be described as a whole by only a few macroscopic parameters, called the thermodynamic variables: the total energy E, volume V, pressure P, temperature T, and so forth,

$$S(E, V, N, \alpha) = k_B \ln \Omega(E, V, N, \alpha) \ , \tag{1}$$

where α represents any other variable which describes the system. The symbol Ω denotes the number of all possible microstates of the system and it is called the thermodynamic probability.

The proportionality constant $k_B = 1.38 \times 10^{23} J/K = 8.62 \times 10^{-5} eV/K$ is one of the fundamental constants of physics, and it is named the Boltzmann constant.

Basics of Statistical Mechanics

Statistical Physics Formulae

Definition of **entropy** S in statistical thermodynamics

$$S(T, V, N) = -k_B \sum_r p_r \ln p_r \ , \quad (2)$$

which, for an isolated system in equilibrium, reduces to Boltzmann's entropy definition for a systerm in equilibrium, as given in (1).

Boltzmann distribution p_r

$$p_r = \frac{1}{Z} \exp(-\beta E_r) \quad (3)$$

where p_r is the probability that a system at temperature T is in the state r with energy E_r.

Partition function Z

$$Z = \sum_r \exp(-\beta E_r) \quad (4)$$

where the summation is over all microstates of the system.

The parameter β is called the temperature parameter,

$$\beta = \frac{1}{k_B T} \tag{5}$$

Mean energy $\overline{E} = \langle E \rangle$ of a system at temperature T is defined as

$$\langle E \rangle = \overline{E} = \sum_r E_r \ln p_r = -\frac{\partial \ln Z}{\partial \beta} = -\frac{1}{Z}\frac{\partial Z}{\partial \beta} \tag{6}$$

Helmholtz free energy F

$$F(T, V, N) = -k_B T \ln Z(T, V, N) \tag{7}$$

$$F = \langle E \rangle - TS \tag{8}$$

$$S(T, V, N) = \frac{\langle E \rangle}{T} - k_B \ln Z \tag{9}$$

Energy dissipation $\langle(\Delta E)^2\rangle$ of a system at temperature T is defined as

$$\langle(\Delta E)^2\rangle = \frac{\partial^2 \ln Z}{\partial \beta^2} = -\frac{\partial \langle E \rangle}{\partial \beta} \tag{10}$$

Heat capacity at constant volume

$$C_V = \frac{\partial \langle E \rangle}{\partial \beta} = \frac{1}{k_B T}\langle(\Delta E)^2\rangle \tag{11}$$

Shannon Entropy

Shannon Entropy

In information theory, the entropy of a random variable is the average level of "information", "surprise", or "uncertainty" inherent to the variable's possible outcomes. Given a discrete random variable X which takes values in the alphabet $\mathcal{X}$ and is distributed according to $p\colon \mathcal{X} \to [0,1]$ the entropy is,

$$H(X) \equiv -\sum_{x\in\mathcal{X}} p(x)\log_a p(x) \ ,$$

where the sum is taken over the variable's possible values. The choice of base a for the logarithm, varies for different applications. Base 2 gives the unit of bits (or "shannons"), while base e for natural logarithms ($\ln x$) gives "natural units" nat, and base 10 gives units of "dits", "bans", or "hartleys". An equivalent definition of entropy is the expected value of the self-information of a variable.

The concept of information entropy was introduced by Claude Shannon in his 1948 paper entitled "A Mathematical Theory of Communication" and is also referred to as Shannon entropy. Shannon's theory defines a data communication system composed of three elements: a source of data, a communication channel, and a receiver.

The "fundamental problem of communication" – as expressed by Shannon – is for the receiver to be able to identify what data was generated by the source, based on the signal it receives through the channel. Shannon considered various ways to encode, compress, and transmit messages from a data source, and proved in his famous source coding theorem that the entropy represents an absolute mathematical limit on how well data from the source can be losslessly compressed onto a perfectly noiseless channel. Shannon strengthened this result considerably for noisy channels in his noisy-channel coding theorem.

As it was mentioned in the introduction already, when Shannon first derived his famous formula for information, he asked von Neumann what he should call it and von Neumann replied “You should call it entropy for two reasons: first because that is what the formula is in statistical mechanises but second and more important, as nobody knows what entropy is, whenever you use the term you will always be at an advantage!!!

Entropy in information theory is directly analogous to the entropy in statistical thermodynamics. The analogy results when the values of the random variable designate energies of microstates, so Gibbs formula for the entropy is formally identical to Shannon’s formula. Entropy has relevance to other areas of mathematics such as combinatorics and machine learning. The definition can be derived from a set of axioms establishing that entropy should be a measure of how informative the average outcome of a variable is. For a continuous random variable, differential entropy is analogous to entropy.

Renyi Entropy

Renyi Entropy

- In information theory, the Rényi entropy is a quantity that generalizes various notions of entropy, including Hartley entropy, Shannon entropy, collision entropy, and min-entropy.
- The Rényi entropy is named after Alfréd Rényi, who looked for the most general way to quantify information while preserving additivity for independent events. In the context of fractal dimension estimation, the Rényi entropy forms the basis of the concept of generalized dimensions.
- The Rényi entropy is important in ecology and statistics as index of diversity.
- The Rényi entropy is also important in quantum information, where it can be used as a measure of entanglement. In the Heisenberg XY spin chain model, the Rényi entropy as a function of α can be calculated explicitly because it is an automorphic function with respect to a particular subgroup of the modular group.
- In theoretical computer science, the min-entropy is used in the context of randomness extractors.

The Rényi entropy of order α, where $0 < \alpha < \infty$ and $\alpha \neq 1$, is defined as,

$$H_\alpha(X) = \frac{1}{1-\alpha} \log \left(\sum_{i=1}^{n} p_i^\alpha \right) .$$

It is further defined at $\alpha = 0, 1, \infty$ as

$$H_\alpha(X) = \lim_{\gamma \to \alpha} H_\gamma(X) .$$

Here, X, is a discrete random variable with possible outcomes in the set $\mathcal{A} = \{x_1, x_2, ..., x_n\}$ and corresponding probabilities $p_i \doteq \Pr(X = x_i)$ for $i = 1, \ldots, n$

The resulting unit of information is determined by the base of the logarithm, e.g. shannon for base 2, or natural for base e where e is the Euler mathematical constant for natural logarithms. If the probabilities are $p_i = 1/n <$ for all $i = 1, \ldots, n$, then all the Rényi entropies of the distribution are equal.

Tsallis Entropy

Tsallis Entropy

Tsallis Entropy [3] is defined through the expression,

$$S_q(p_i) = k\frac{1}{q-1}\left(1 - \sum_i p_i^q\right) ,$$

where, q is a real parameter sometimes called "entropic-index" and k a positive constant.
In the limit as $q \to 1$, the usual Boltzmann–Gibbs entropy is recovered, namely

$$S_{\mathsf{BG}} = S_1(p) = -k\sum_i p_i \ln p_i$$

Ishihara-2019: Tsallis-Linear Sigma Model
Petropoulos-2004: Linear Sigma Model

[3] Tsallis, C. Possible generalization of Boltzmann-Gibbs statistics. J Stat Phys 52, 479–487 (1988). https://doi.org/10.1007/BF01016429

Non-additivity

Given two independent systems A and B, for which the joint probability density function satisfies,

$$p(A, B) = p(A)p(B) \ ,$$

then, the Tsallis entropy of this system satisfies,

$$S_q(A, B) = S_q(A) + S_q(B) + (1 - q)S_q(A)S_q(B) \ .$$

From this result, it is evident that the parameter $(1-q)$ is a measure of the departure from additivity. In the limit when the parameter $q = 1$, we find

$$S(A, B) = S(A) + S(B) \ ,$$

which is what is expected for an additive system. This property is sometimes referred to as "pseudo-additivity".

Kaniadakis Entropy

Kaniadakis Entropy

The Kaniadakis statistics is based on the Kaniadakis κ–entropy, which is defined through:

$$S_\kappa(p) = -\sum_i p_i \ln_\kappa(p_i) = \sum_i p_i \ln_\kappa\left(\frac{1}{p_i}\right)$$

where $p = p_i = p(x_i)$, $x \in \mathbb{R}$ and $i = 1, 2, ..., N$

As usual, normalization of probability demands,

$$\sum_i p_i = 1 \ ,$$

while κ, where $0 \leq |\kappa| < 1$ is called the entropic index.

The Kaniadakis κ–entropy is thermodynamically and Lesche stable and obeys the Shannon-Khinchin axioms of continuity, maximality, generalized additivity and expandability.

Starting with three influential papers twenty years ago Giorgio Kaniadakis [4] [5] [6] has pioneered the extension of Boltzmann's Stosszahlansatz (molecular chaos hypothesis) in the framework of special relativity by proposing a new entropy, which emerged as the relativistic generalization of the Boltzmann–Shannon entropy. The Kaniadakis entropy generates power-law tailed statistical distributions, which in the classical limit reduce to the Maxwell–Boltzmann exponential distribution.

This new entropy, also known as κ-entropy or κ-deformed entropy, is considered as one of the most viable candidates for explaining the experimentally observed power-law tailed statistical distributions in various physical, natural, and artificial, complex systems.

[4] Physica A 296, 405 (2001)

[5] Phys. Rev. E 66, 056125 (2002)

[6] Phys. Rev. E 72, 036108 (2005)

Following the introduction of the Kaniadakis entropy, more than 150 statistical physics papers contributed by more than 200 scientists have been published on the subject. Relevant advances have been made in the physical foundations and mathematical formalism of the theory, as well as its applications in statistical physics and thermodynamics, quantum statistics, quantum theory, plasma physics, nuclear fission, particle physics, astrophysics and cosmology, seismology and geophysics, waveform inversion, image processing, machine learning, networks, information theory and statistical sciences, fractal theory, genomics, biophysics, economics, finance, social sciences, and complex systems, among other topics.

The study of the Kaniadakis entropy and related functions is emerging as a rapidly developing research field which attracts a steadily increasing number of researchers from different countries and spans an ever-increasing domain of applications.

Barrow Entropy

Barrow Entropy

In Cosmology horizon entropy maximization has been examined assuming the semi-classical Bekenstein-Hawking area law for the horizon entropy,

$$S = \left(\frac{A}{A_0}\right)$$

where A and denote the surface area of the Universe and Planck area, respectively. Barrow entropy appears in the form:

$$S = \left(\frac{A}{A_0}\right)^{1+\Delta/2}$$

where $0 \leq \Delta \leq 1$ quantifies departure from Bekenstein-Hawking area law. In particular, $\Delta = 0$ gives the standard limit, while $\Delta = 1$ corresponds to its maximal deformation.

It has attracted much attention recently and there many research papers in cosmology dealing with this idea.

Prigogine ideas about entropy

Prigogine: "La Fin des Certitudes"

There's not space here to discuss Prigogine's ideas about entropy, arrow of time, systems out of equilibrium, not even to touch it ...

It should be many talks ...!!!

Ilya Prigogine, "End of Certainty", an intriguing book ...!!!!!

$EYXAPI\Sigma T\Omega \;\; \Pi O\Lambda Y!!$

Thank you very much!!

Obrigado Muito!!

Muchas Gracias!!